Two problems in real and complex integral geometry

A. M. Naveira
Departamento de Geometría y Topología
Facultad de Matematicas
46100-Burjassot, Valencia, Spain
naveira@uv.es

Abstract

In this article we state two problems related with Integral Geometry. In the first we try to obtain analytical inequalities which become equivalent to the inequalities among the integrals of the mean curvatures of a hypersurface in the euclidean space. The seond problem is related to the Complex Integral Geometry. We try to analyse the "complex cross-sectional measures" of a convex body contained in the complex euclidean space.

Mathematics Subject Classifications (2000): 53C65.

Key words: Isoperimetric inequality, cross sectional measures, mixed volume, Sobolev spaces, complex integral geometry.

1 In memoriam of Luis Antonio Santaló

In order to render a small homage to Prof. L. A. Santaló, I will state two problems related to Integral Geometry. First, however, I would like to recall in a few words how our scientific and personal relationship began and flourished.

During my PhD years at the University of Santiago my friend and advisor Prof. Enrique Vidal Abascal always spoke with enormous respect and affection of a spanish mathematician then living in Argentina: Prof. Luis Antonio Santaló Sors. Prof. Vidal had consulted him many times over several ideas arising from the study of his articles.

In fact, as he was fond of telling me, Prof. Vidal, while studying Bieberbach's book in the middle of the forties, became more and more interesting in the dawning problems of the Integral Geometry. In particular, he was interested in Steiner's formula for spaces with constant mean curvature, the study of invariant measures in homogeneous spaces and the integral invariants. Prof. Vidal went on to publish some papers on these problems.

As all homogeneous spaces define a foliation in a natural way, in the following years, Prof. Vidal turned to the study and analysis of foliated manifolds in general. Along these lines and under his supervision I wrote my thesis in Santiago during the year 1968. At that time, in Spain the influence of Santaló spread to other mathematicians, amongst which I would like to point out Prof. E. García-Rodeja and Prof. J. Sancho de San Román.

In 1967 an international congress on Differential Geometry was held in Santiago. At this congress, Prof. Santaló was specially invited. Then and there I had the pleasure and honor to meet him and appreciate his extraordinary human qualities. However, due to my sparse background in the area, I could not fully understand nor admire the richness of his talk in which he defined the absolute total curvatures of a closed subset of the euclidean space. This definition became important in a great part of his later work concerning both Integral Geometry and its applications to other scientific branches.

Eleven years later, when Prof. Vidal was retiring, a new congress on Differential Geometry was held in Santiago and Prof. Santaló was invited. On this occasion, we had the opportunity to discuss and exchange points of view on a wide range of mathematical topics. Just after this congress, Prof. Santaló visited the University of Valencia giving the many young researchers in Differential Geometry the stimulating opportunity to discuss with him several aspects of recent lines of research.

In the early eighties, along with my PhD Salvador Segura Gomis, I became more deeply interested in Integral Geometry, when we observed that the Isoperimetric Problem was related both to Integral Geometry and to Analysis.

In November 91, Prof. Santaló gave a two week course in the *"Chair of Contemporaneous Thinking Ferrater Mora"* of Girona. By his indications, several professors linked both to spanish and foreingn universities were invited to take part in the course.

There I met, amongst others, Fernando Affentranger and Luis Cruz-Orive. This was a unique event in my academic life since it allowed me to establish with Prof. Santaló and his family an unforgettable scientific and personal relationship.

At that conference I was accompanied by my PhD student Ximo Gual Arnau who was also interested in Integral Geometry. Afterwards, Gual Arnau and Cruz-Orive have published several papers together applying methods of Integral Geometry and Stereology to Tomography with quite useful results in Biomedicine.

In August 1991 I was invited to give a talk at the Mathematics Department of the University of Buenos Aires. Since Prof. Santaló was present at in the talk, I found this a unique opportunity to state before part of the argentinean mathematical community his extraordinary qualities, his exceptional scientific achievements and the influence he has had over many spanish mathematicians during the last sixty years.

Prof. Santaló was always very aware of the surrounding problems. Therefore his work was carried out in several fields: scientific investigation in Differential Geometry, mainly in Integral Geometry and Theoretical Physics; his concern over mathematical education at all levels, but specially in High School and University; and his wish to bring together Mathematics and Society making Mathematics understandable to all scientific community, thus becoming a scientific popularizer.

Prof. Santaló had a clear and fertile mind, with an uncommon intelligence and intuition. As for his humane aspects, he was a kindhearted person always ready to aid and collaborate within his limits with all the scientific spanish community and a greater length with the mathematical community.

He constantly tried to expand the culture to the whole of society and along these lines he once reminded us *"We teach for the good, for the truth and in order to know and understand the Universe"*.

These traits along with his scientific work make him one of the most renowned spanish scientist of the last century. For political reasons, he spent most of his life in Argentina. There he settled down with the help of his previous teachers Prof. Rey Pastor and Prof. Terradas. This exile can be considered one of the most influential and saddest facts affecting the development of spanish mathematics at that time. However, his intelligence, his dedication, his capacity for working long hours, his love for Spain and Catalonia, have undoubtedly contributed to the downing of the mathematical development in our country.

While analysing Prof. Santaló work, we can outline some of his fundamental characteristics: his *"abstraction power"*, a brilliant geometric intuition and an extraordinary ability to communicate. Therefore it is not strange that his work has had a deep repercussion both in the scientific community and in the society at large.

People like Prof. Santaló will always be necessary for the development of mathematics, its applications and its generalization.

For his exceptional contributions, I would like to give him my deepest and sincere gratitude, respect, affection and admiration. I would also like to thank the *"Chair Lluis Santaló"* from the University of Girona that has rendered me the opportunity to add to his well deserved homage.

2 Introduction

The concept of mixed volume or integral cross-sectional measure (quermassintegrale) is well known in the specialized mathematical literature [BZ, S1]. It refers to a family of intrinsic measures, among which there are a number of inequalities: for instance, "The classical Isoperimetric Inequality". A beautiful proof of the equivalence of its geometric and analytic formulations can be found in [B]. In the first part of this note we pose the problem of generalizing the equivalence to other isoperimetric-like geometric inequalities formulated in terms of the different integral cross-sectional measures.

We can say that the history of Integral Geometry begins with Buffon's needle problem at the end of XVIII century. It grew up in an incipient way throughout the XIX century, but most of its development took place in the XX century, basically with the work of Blaschke and his school, and later with Santaló and Chern, among others.

Santaló's book has been a basic tool for the study of Integral Geometry in the last decades of the XX century. It is, and will continue to be a very important reference for all researchers in this field. For convex sets in the euclidean space, Santaló defines the real integral cross-sectional-measures as an average integral on the real Grassmann manifold of real subspaces through the origin. He also considers that a study in depth of the Complex Integral Geometry could be interesting [S1, p. 338]. In [NG, N], many of the properties of densities in real spaces has been extended to the complex case. Here it is also possible to define, by means of an integration in the corresponding complex Grassmann manifold, the concept of complex integral cross-sectional measures. In the second part of this note we pose the following question: How is the representation of the complex cross-sectional measures in terms of the integrals of the generalized mean curvatures?

3 Inequalities of isoperimetric type

It is well known that $A+B = \{a+b \mid a \in A, b \in B\}$ denotes the vector sum (Minkowski sum) of the subsets A and B of Euclidean space $I\!R^n$, while $\lambda A = \{\lambda a \mid a \in A\}$ is the result of the homothethy of A with coefficient λ. We consider only non-empty convex compact subsets of the space $I\!R^n$, often without saying it explicitly.

Theorem of Minkowski [BZ, p. 136]. The volume of the linear combination of non-empty compact sets $K_1, ..., K_s$ ($s \neq n$, in general) with nonnegative coefficients $\lambda_1, ..., \lambda_s$ is a homogeneous polynomial of degree n with respect to $\lambda_1, ..., \lambda_s$:

$$V(\Sigma_{i=1,\dots,s}\lambda_i K_i) = \Sigma_{i_1=1,\dots,s} \dots \Sigma_{i_s=1,\dots,s} V(K_{i_1}, ..., K_{i_s})\lambda_{i_1}...\lambda_{i_s}.$$

It follows from the theorem that these coefficients depend only on those $K_1, ..., K_s$.

The classical definition of *mixed volume* of non-empty convex compact sets $K_1, ..., K_n$ in $I\!R^n$ (they are not necessarily distinct and their order plays no role) according to Minkowski is the following: this volume is the coefficient $V(K_1, ..., K_n)$ in the decomposition of the Minkowski theorem involving these sets.

In geometry one usually considers only those mixed volumes for which, only two (rarely three) of the K_i differ. Most often one considers the so-called m-th *integral cross-sectional measures*

$$V_m(K) = V(K, \dots (m) \dots, K, D, \dots (n-m) \dots, D)$$

where K is a non-empty convex compact set, while D is the closed unit ball in $I\!R^n$.

Their particular cases (with the appropriate normalization) are the volume $V(K)$ and the $(n-1)$-dimensional boundary area $S(K) : V(K) = V_n(K)$, and $S(K) = nV_{n-1}(K)$.

It is well known the following:

Theorem [S1]. The value of $V_m(K)$ coincides (up to a factor depending only on m and n) with the mean integrated value of m-dimensional volumes of the projections of K on all possible m-dimensional subspaces $\nu = I\!R^m \subset I\!R^n$. To be more precise,

$$V_m(K) = (v_n/v_m)\int_{G_{m,n-m}} V(\pi_\nu(K))d\omega(\nu),$$

where $G_{m,n-m}$ is the Grassmann manifold of m-dimensional subspaces in $I\!R^n$; $\pi_\nu(K)$ is the orthogonal projection of K on $\nu \in G_{m,n-m}$; ω is the Haar measure on $G_{m,n-m}$, and $\omega(G_{m,n-m}) = 1$; and v_i denotes the volume of the i-dimensional euclidean ball.

This theorem justifies the term *integral cross-sectional measures.*

Theorem [BZ]. The Alexandrov-Fenchel Inequality said that

$$V^2(K_1, K_2, ..., K_n) \geq V(K_1, K_1, K_3, ..., K_n)V(K_2, K_2, K_3, ..., K_n).$$

As particular cases of the Alexandrov-Fenchel inequality, we can consider a whole series of well-known geometric inequalities:

i) The classical Isoperimetric Inequality

$$S^n(K) \geqslant n^n v_n V^{n-1}(K),$$

where K is a non-empty convex-compact set in $I\!R^n$; $S(K) = n\ V^{n-1}(K)$ is the $(n-1)$-dimensional area of ∂K; $V(K)$ is its volume; and v_n the volume of the unit ball D in $I\!R^n$. This inequality is known as the first Minkowski Inequality.

ii) The second Minkowski Inequality

$$V_1^n(K) \geq v_n^{n-1}V(K).$$

iii) The Minkowski Quadratic Inequalities

$$V_{n-1}^2(K) \geq V_{n-2}(K)V(K) \qquad V_1^2(K) \geq v_n V_2(K).$$

iv) The Favard Inequalities, which generalizes the Minkowski quadratic inequalities

$$V_i^2(K) \geq V_{i-1}(K)V_{i+1}(K).$$

v) The generalized Minkowski Inequalities, which generalizes the first and the second Minkowski Inequalities

$$V_i^n(K) \geq v_n^{n-i}V^i(K).$$

vi) The Alexandrov Inequality

$$V_j^i(K) \geq v_n^{i-j}V_i^j(K),$$

for $j \leq i$.

It is well known that for all these inequalities the equality holds if and only if K is the euclidian ball.

It is well known in the literature the Steiner decomposition for the volume of a parallel body in function of the volume, surface area and the integrals of mean curvature of the primitive body [BZ, S1]. The corresponding formula is

$$V_n(K+rD) = V_n(K) + nV_{n-1}(K)r + C_{n,2}V_{n-2}(K)r^2 + ... + C_{n,n-1}V_1(K)r^{n-1} + v_n r^n.$$

If in the above formula we substitute $r = r_1 + r_2$, then carry out this decomposition consecutively for r_1 and r_2 and equate the coefficients, then we obtain the Steiner decomposition for m-th integral cross-sectional measures

$$V_m(K+rD) = V_m(K) + mV_{m-1}(K)r + C_{m,2}V_{m-2}(K)r^2 + ... + C_{m,m-1}V_1(K)r^{m-1} + v_n r^m.$$

We can find also these formulae in [S1, p. 221], using there the theory of integral geometry.

Corollary [BZ]. From the above theorem we have the Brunn-Minkowski Inequality for integral cross-sectional measures. That is,

$$V_m^{1/m}(K+L) \geq V_m^{1/m}(K) + V_m^{1/m}(L).$$

In [B] we can find a beautiful proof of the Isoperimetric Inequality, using the Brunn-Minkowski Inequality for V_n.

We would like to recall the following result because of its importance:

Theorem [BZ, p. 139; and S1, p. 222]. For a non-empty convex K with C^2-smooth boundary ∂K, with all the principal curvatures k_j non zero, the value $V_m(K)$ coincides, up a factor (depending only on m and n), with the integral over the spherical map of the m-th elementary symmetric function of the principal radii of curvature of ∂K of the $(n-m-1)$-th elementary symmetric function of the principal curvatures k_j. To be more precise,

$$V_m(K) = (v_n/\omega_{n-1}C_{n-1,m}) \int_{S^{n-1}} \sum_j R_{j_1}...R_{j_m} d\omega(u),$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$; R_j, $(j = 1, ..., n-1)$ are the principal radii of curvature at a point of ∂K with outer normal $u \in S^{n-1}$; ω is the area on S^{n-1}; $\omega_{n-1} = \omega(S^{n-1})$; and the sum is taken over the $C_{n-1,m}$ various finite sequences $j = (j_1, ..., j_m)$, $1 \leq j_i \leq n-1$. Or

$$V_m(K) = (1/nC_{n-1,m}) \int_{\partial K} \sum_j k_{j_1}...k_{j_{n-m-1}} dF(x),$$

where the k_i, $(i = 1, ..., n-1)$ are the principal curvatures of ∂K at a point $x \in \partial K$; dF is the element of area on ∂K; and the sum is taken over the $C_{n-1,n-m-1}$ various finite sequences $j = (j_1, ..., j_{n-m-1})$, $1 \leq j_i \leq n-1$.

In the following we represent by

$$nV_m(K) = \int_{\partial K} \sigma_{n-m-1}(\partial K) dF(x).$$

That is, σ_{n-m-1} is the mean curvature of order $n-m-1$ of ∂K.

It is well known that there is a connection between isoperimetric inequality and the Sobolev embedding t6heorems. As a particular case of the general situation [A, p. 40] we have the following result of Federer and Fleming or Maz'ya and a proof of it is presented in [B].

Theorem [B]. Let A be a bounded domain in $I\!R^n$ with smooth boundary ∂A. The isoperimetric inequality

$$V_{n-1}^n(\partial A) \geq v_n V_n^{n-1}(A),$$

for every such A, is equivalent to the inequality

$$\left(\int_A \|grad\ f\| dA\right)^n \geq v_n \left(\int_A \|f\|^{n/(n-1)} dA\right)^n,$$

for all $f \in C_0^\infty(A)$.

For the proof of the above result it was basic the fact that the surface area of the level-hypersurfaces were positive decreasing function [B, p. 301].

> **OPEN PROBLEM:** *In the direction of the above Theorem, it seems natural try to obtain analytical inequalities which becomes equivalent to i) - vi).*

4 Complex cross-sectional measures

It is well known that on a Lie group $\mathbf{G}$ it is possible to define a left invariant volume form $d\mathbf{G}$. Let $\mathbf{G}$ be a Lie group of dimension n and let $\mathbf{H}$ be a closed subgroup of $\mathbf{G}$ of dimension $n-m$. The set of left cosets $g\mathbf{H}$, $g \in \mathbf{G}$, is the homogeneous space $\mathbf{G}$ $/\mathbf{H}$, which, as it is known, admits a differentiable manifold structure of dimension m. We want to find the conditions for the existence of a nonzero m-form on $\mathbf{G}/\mathbf{H}$ invariant under $\mathbf{G}$. Such an m-form is called a "*density*" on $\mathbf{G}/\mathbf{H}$ and it gives rise, by integration, to an "*invariant measure*" on $\mathbf{G}/\mathbf{H}$. Since $\mathbf{G}$ acts transitively on $\mathbf{G}/\mathbf{H}$, the invariant density, if it exist, is unique up to a constant factor.

Note that $\mathbf{H}$ and $g\mathbf{H}$ are differentiable submanifolds of $\mathbf{G}$ and $\mathbf{G}$ is foliated by $g\mathbf{H}$. Then we know that $g\mathbf{H}$ are the integral manifolds of a completely integrable Pfaffian system

$$\omega_1 = 0, ..., \omega_m = 0,$$

where ω_i are 1-forms on $\mathbf{G}$. It is easy to prove that

$$d(\mathbf{G}/\mathbf{H}) = \omega_1 \bigwedge ... \bigwedge \omega_m$$

is invariant under $\mathbf{G}$ and, up to a constant factor, it is the unique m-form with this property.

Theorem [S1]. A necessary and sufficient condition for the m-form $d(\mathbf{G}/\mathbf{H})$ to be a density for $\mathbf{G}/\mathbf{H}$ is that its exterior differential vanish. That is,

$$d(d(\mathbf{G}/\mathbf{H})) = 0.$$

In the mathematical literature, in particular in the Santaló's book, is developed the theory of the Real Integral Geometry supported in the above Theorem. Also, Santaló said in his book "*Integral Geometry on complex spaces has not been sufficiently developed and probably deserves further study*". Some results are known about the complex projective space and the unitary group [S2].

Let $\mathbb{C}^n$ denote the space of n-tuples of complex numbers $(z_1, ..., z_n)$. The r-dimensional complex linear spaces of $\mathbb{C}^n$ will be called r-planes. We consider the group of complex motions $\mathcal{M}$, $z' = az + b$, where a is an element of the unitary group $U(n)$ and b is a complex number. The study of $\mathcal{M}$ can be easily done by the method of complex moving frames. Let (p, e_i, e_{i*}) be a complex moving frame. So, applying the standard method we have

$$\omega_i = dp \cdot e_i, \quad \omega_{i*} = dp \cdot e_{i*}, \quad \omega_{ji} = de_i \cdot e_j, \quad \omega_{ji*} = de_{i*} \cdot e_j,$$

$$\omega_{ji} + \omega_{ij} = 0, \quad \omega_{ji*} + \omega_{i*j} = 0, \quad \omega_{j*i*} + \omega_{i*j*} = 0, \quad \omega_{ij*} + \omega_{i*j} = 0, \quad \omega_{ji} = \omega_{j*i*},$$

The complex rotations about a point can be identified with the unitary group $U(n)$ which is compact and has an invariant measure which was determinated by Santaló [S2]. Since the complex translations had also an invariant density, we have an invariant density for the group of the complex motions.

Then, we can generalize to the Complex Integral Geometry on $\mathbb{C}^n$ all results known relatives to densities in the Real Integral Geometry.

The complex Grassmann manifold $G^c_{m,n-m}$ of the complex m-planes trought the origin is well defined as

$$G^c_{m,n-m} = U(n)/U(m) \times U(n-m),$$

and we can determine its volume.

We can also consider a convex body

$$Q^{2n-1} \subset \mathbb{R}^{2n} \equiv \mathbb{C}^n.$$

I assume that ∂Q is sufficiently smooth. Then, the m-th *complex integral cross-sectional measure* $V^c_m(Q)$ may be defined as the mean integral value of the m-dimensional volumes of the projections of Q on m-dimensional complex subspaces. Explicitly, we write

$$V^c_m(Q) = \int_{G^c_{m,n-m}} \mathcal{H}^m(\pi_\nu(Q) d\mu(\nu),$$

where $G^c_{m,n-m}$ is the complex Grassmann manifold of m-dimensional complex subspaces in $\mathbb{C}^n$; $\pi_\nu(Q)$ is the orthogonal projection of Q on the subspace $\nu \in G^c_{m,n-m}$; $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure in $\mathbb{C}^n$; and μ denotes the normalized Haar measure on $G^c_{m,n-m}$.

In the real case has been shown as the m-th integral cross-sectional measures could be represented by the integral of the symmetric functions of the principal curvatures.

> **OPEN PROBLEM:** *How is the representation of $V^c_m(Q)$ in function of the integrals of the generalized mean curvatures?*

Acknowledgement

This work is partially supported by DGI grant number BFM2001-3548.

References

[A] Aubin, T. *Non linear Analysis on Manifolds. Monge-Ampère equations*, Springer-Verlag (1982).

[B] Bombieri, E. *An Introduction to Minimal Currents and Parametric Variational Problems*, Math. Rep. V, 2, Part 3 (1985).

[BZ] Burago, Y. D. and Zalgaller, V. A. *Geometrical Inequalities*, Springer-Verlag (1988).

[NG] Naveira, A. M. and García, F. *Some results of integral geometry for density of linear subspaces of $\mathbb{C}^n$*, Rendiconti di Mathematica di Roma 12 (1992) 921 - 935.

[N] Naveira, A. M. *Some remarks about integral geometry in the complex space $\mathbb{C}^n$*, Rendiconti di Mathematica di Roma 13 (1993) 331 - 346.

[O] Ossermann, R. *The Isoperimetric Inequality*, Bull. Amer. Math. Soc. 84 (1978) 1182 - 1238.

[S1] Santaló, L. A. *Integral Geometry and Geometric Probability*, Addison-Wesley (1977).

[S2] Santaló, L. A. *Integral Geometry in Hermitian Spaces*, Amer. J. Math. 74 (1952) 423 - 434.

[Sch] Schneider, R. *Convex bodies: The Brunn-Minkowski-Theory*, Enc. of Math. Appl. 44 Cambridge Univ. Press (1993).

[T1] Trudinger, N. S. *Maximum principles for curvature quotient equations*, J. Math. Sc. Univ. Tokyo 1 (1994) 551 - 565.

[T2] Trudinger, N. S. *On new isoperimetric inequalities and symmetrization*, J. Reine Angew. Math. 488 (1997) 203 - 220.

[T3] Trudinger, N. S. *Isoperimetric Inequalities for Quermassintegrals* Ann. Inst. H. Poincaré 11 (1994) 411 - 425.

[W] Warner, F. W *Foundations of Differentiable Manifolds*, Scott Foresman (1970).