



## Complete systems of inequalities

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### Abstract

With this contribution we summarize the known results and the main tools concerning *complete systems of inequalities* for convex sets. The first problem of this type was posed by Blaschke in 1916, although it was Santaló who, in 1961, studied and developed this kind of problems.

### 1 The origin of this concept: Blaschke's Problem

One of the oldest problems in global geometry is the well known isoperimetric problem, i.e. finding among all planar domains with fixed perimeter the domain enclosing maximum area. The solution to this problem is the *isoperimetric inequality*:

“Let  $K$  be a planar compact convex set, and let  $A(K)$  and  $p(K)$  be, respectively, its area and its perimeter; then

$$p(K)^2 \geq 4\pi A(K)$$

where equality sign is attained if and only if  $K$  is a circle.”

The result was known by the ancient Greeks but its rigorous proof was not available until the end of the XIX century.

An interesting question is whether the isoperimetric inequality is not only a necessary but also a sufficient condition for two positive numbers  $A_0$ ,  $p_0$  to be the area and the perimeter of some convex body in the plane. The answer turns out to be positive as it can be seen considering, for instance, a family of ellipses.

The area and the perimeter of a planar convex set  $K$  are not only very natural measures which give interesting geometric information about the set  $K$ , but also a pair of geometric magnitudes playing an important role in the theory of convex sets, the so called 2-dimensional cross-sectional measures. So, in 1916 Blaschke asked the same question for the 3-dimensional cross-sectional measures of a 3-dimensional convex body  $K$ : the volume  $V(K)$ , the surface area  $S(K)$  and the integral mean curvature  $M(K)$ .

The inequalities relating the above magnitudes are the isoperimetric inequality for planar convex sets of the euclidean space  $E^3$

$$2M(K)^2 \geq \pi^3 S(K), \quad \text{where } V(K) = 0, \quad (1)$$

and Minkowski's inequalities,

$$S(K)^2 \geq 3V(K)M(K), \quad (2)$$

$$M(K)^2 \geq 4\pi S(K), \quad (3)$$

$$M(K)^3 \geq 48\pi^2 V(K). \quad (4)$$

So, the problem raised by Blaschke was the following:

Are the isoperimetric inequality and Minkowski's inequalities not only necessary conditions but also sufficient conditions for three positive numbers  $V_0$ ,  $S_0$  and  $M_0$  to be the volume, the surface area and the integral mean curvature of a 3-dimensional convex body?

The answer to Blaschke's question happened to be much more difficult and in fact it is still an open problem. However, Blaschke developed a very useful technique known as *Blaschke's diagram*: he defined a map from the family of all compact convex 3-dimensional sets by means of a convenient choice of coordinates, into a compact region of the unit square in the plane. More precisely, he wrote

$$x = \frac{4\pi S}{M^2} \quad \text{and} \quad y = \frac{48\pi^2 V}{M^3}.$$

Inequalities (3) and (4) guaranteed that  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ ; using the coordinates  $x$  and  $y$  defined above, inequalities (1) and (2) represent two arcs of the boundary of Blaschke's diagram, defined by the parabola  $y = x^2$  and the straight line segment  $y = 0$  with  $0 \leq x \leq 8/\pi^2$ ; on the other hand, Blaschke's Selection Theorem guarantees that the image of Blaschke's map is a compact set.

So, since the arcs of the boundary defined above do not close the whole diagram, we can conclude that at least one inequality is missing (see Figure 1).

Despite all the work done after Blaschke by Hadwiger, Bieri, Groemer, Sangwine-Yager and others, the problem is still open.

Then, fixing any class of figures and any finite set of numerical characteristics of these figures, Hadwiger (Hadwiger, 1948) defined a *complete system of inequalities* as a system of inequalities relating all these characteristics such that for any set of numbers satisfying the inequalities, a figure with these values of the characteristics exists in the given class.

So, Blaschke's problem could be rewritten as the problem of finding a complete system of inequalities for  $V$ ,  $S$  and  $M$  in the class of 3-dimensional convex sets. But, as Blaschke's problem remained open, there was no non-trivial example of these systems.

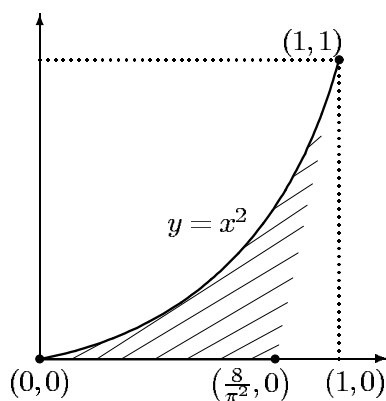


Figure 1: Blaschke's Diagram.

## 2 Santaló's Problem

It was Santaló who, in 1961, obtained the first examples applying Blaschke's technique to the six classic geometric magnitudes of a planar convex set (Santaló, 1961):

Let  $K$  be a planar compact convex set, with area  $A(K)$ , perimeter  $p(K)$ , diameter  $d(K)$ , minimal width  $\omega(K)$ , circumradius  $R(K)$  and inradius  $r(K)$ . He gave the following list of the known inequalities involving two functionals, determining also the extremal sets:

Table 1: Inequalities for pairs of magnitudes.

Cases	Inequalities	Equality	Cases	Inequalities	Equality
$(A, p)$	$4\pi A \leq p^2$	Circle	$(p, r)$	$2\pi r \leq p$	Circle
$(A, d)$	$4A \leq \pi d^2$	Circle	$(d, \omega)$	$\omega \leq d$	Constant Width Sets
$(A, \omega)$	$\omega^2 \leq \sqrt{3}A$	Equilateral Triangle	$(d, R)$	$d \leq 2R$	Many Figures
$(A, R)$	$A \leq \pi R^2$	Circle		$\sqrt{3}R \leq d$	Yamanouti Sets
$(A, r)$	$\pi r^2 \leq A$	Circle	$(d, r)$	$2r \leq d$	Circle
$(p, d)$	$p \leq \pi d$	Constant Width Sets	$(\omega, R)$	$\omega \leq 2R$	Circle
	$2d \leq p$	Line Segments	$(\omega, r)$	$\omega \leq 3r$	Equilateral Triangle
$(p, \omega)$	$\pi\omega \leq p$	Constant Width Sets		$2r \leq \omega$	Many Figures
$(p, R)$	$p \leq 2\pi R$	Circle	$(r, R)$	$r \leq R$	Circle
	$4R \leq L$	Line Segments			

This list determines all the complete systems of inequalities for the fifteen possible cases of pairs chosen among the six classical geometric magnitudes.

Then, following Blaschke's idea, he considered all the twenty triples of these magnitudes and for each of these cases he tried to find complete systems of inequalities.

First, Santaló gave the following list of the known inequalities involving three functionals, two of them were obtained by himself, determining also the extremal sets:

Table 2: Inequalities for triples of magnitudes.

Inequalities	Condition	Equality
$8\phi A \leq p(p - 2d \cos \phi), \quad p \sin \phi = 2d\phi$		Symmetric Lens
$4A \leq 2\omega p - \pi\omega^2$		Sausages
$2A \geq \omega p - \sqrt{3}\omega^2 \sec^2 \phi, \quad 6\omega(\tan \phi - \phi) = p - \pi\omega$	$\pi\omega \leq p \leq 2\sqrt{3}\omega$	Yamanouti Sets
$16A^2\omega^3 \geq p(p\omega - 4A)^2(8A - \omega p)$	$2\sqrt{3}\omega \leq p$	Isosceles Triangles
$8\phi A \leq p(p - 4R \cos \phi), \quad p \sin \phi = 4R\phi$		Symmetric Lens
$A \leq r(p - \pi r)$		Sausages
$2A \geq pr$		Cap-Bodies
$2A \leq \omega\sqrt{d^2 - \omega^2} + d^2 \arcsin(\omega/d)$		Circular Symmetric Slices
$2A \geq \omega d$	$\omega \leq \sqrt{3}/2d$	Triangles
$A \geq 3\omega [\sqrt{d^2 - \omega^2} + \omega (\arcsin(\omega/d) - \pi/3)] - \sqrt{3}/2d^2$	$\sqrt{3}/2d \leq \omega \leq d$	Yamanouti Sets
$p \leq 2\sqrt{d^2 - \omega^2} + 2d \arcsin(\omega/d)$		Circular Symmetric Slices
$p \geq 2\sqrt{d^2 - \omega^2} + 2\omega \arcsin(\omega/d)$		Cap-Bodies
$r \leq \sqrt{4R^2 - d^2} - 2R + d^2/(2R)$		Isosceles Triangles
$d \geq R + r$		Constant Width Sets
$\omega \leq R + r$		Constant Width Sets

Santaló solved six of the twenty possible cases:  $(A, p, \omega)$ ,  $(A, p, R)$ ,  $(A, p, r)$ ,  $(A, d, \omega)$ ,  $(p, d, \omega)$  and  $(d, R, r)$ .

As an example let us describe the case  $(A, d, \omega)$ .

### 3 An example: The complete system of inequalities for $(A, d, \omega)$

For the area, the diameter, and the width of a compact convex set  $K$ , the relationships between pairs of these geometric measures are:

$$4A \leq \pi d^2 \quad \text{Equality for the circle} \quad (5)$$

$$\omega^2 \leq \sqrt{3}A \quad \text{Equality for the equilateral triangle} \quad (6)$$

$$\omega \leq d \quad \text{Equality for sets of constant width} \quad (7)$$

And the relationships between three of those measures are:

$$2A \leq \omega\sqrt{d^2 - \omega^2} + d^2 \arcsin \frac{\omega}{d}, \quad (8)$$

with equality for the circular symmetric slice,

$$2A \geq \omega d, \quad \text{if } 2\omega \leq \sqrt{3}d, \quad (9)$$

with equality for the triangles, and

$$A \geq 3\omega \left[ \sqrt{d^2 - \omega^2} + \omega \left( \arcsin \frac{\omega}{d} - \frac{\pi}{3} \right) \right] - \frac{\sqrt{3}}{2}d^2, \quad \text{if } 2\omega \geq \sqrt{3}d, \quad (10)$$

with equality for the Yamanouti sets.

Let

$$x = \frac{\omega}{d} \quad \text{and} \quad y = \frac{4A}{\pi d^2}.$$

Clearly, from (7) and (5), it holds  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ .

From inequality (8) we obtain

$$y \leq \frac{2}{\pi} \left( x\sqrt{1-x^2} + \arcsin x \right) \quad \text{for all } 0 \leq x \leq 1.$$

Then, the curve

$$y = \frac{2}{\pi} \left( x\sqrt{1-x^2} + \arcsin x \right)$$

determines the upper part of the boundary of the diagram  $\mathcal{D}$ . This curve connects point  $O = (0, 0)$  (corresponding to line segments) with point  $C = (1, 1)$  (corresponding to the circle), and the circular symmetric slices are mapped to the points of this curve (see Figure 2).

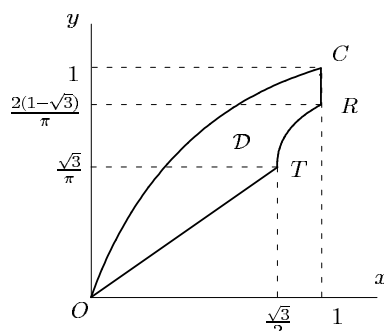


Figure 2: Blaschke-Santaló Diagram for the case  $(A, d, \omega)$ .

The lower part of the boundary is determined by two curves obtained from inequalities (9) and (10).

The first one is the line segment

$$y = \frac{2}{\pi} x \quad \text{where} \quad 0 \leq x \leq \frac{\sqrt{3}}{2},$$

which joins  $O$  and  $T = (\sqrt{3}/2, \sqrt{3}/\pi)$  (equilateral triangle); its points represent the family of the triangles.

The second curve is

$$y = \frac{12}{\pi} x \left[ \sqrt{1-x^2} + x \left( \arcsin x - \frac{\pi}{3} \right) \right] - 2 \frac{\sqrt{3}}{\pi} \quad \text{where} \quad \frac{\sqrt{3}}{2} \leq x \leq 1.$$

This curve completes the lower part of the boundary, from the point  $T$  to  $R = (1, 2(1 - \sqrt{3}/\pi))$  (Reuleaux triangle). The Yamanouti sets are mapped to the points of this curve (see Figure 2).

Finally, inequality (7) gives  $x \leq 1$ , and the boundary of  $\mathcal{D}$  is closed with the line segment  $\overline{RC}$  ( $x = 1$ ) which represents the sets of constant width, from the Reuleaux triangle (minimum area) to the circle (maximum area).

Now, before assuring that the above inequalities form a complete system, we have to see that the domain  $\mathcal{D}$  is simply connected, i.e., there are convex sets which are mapped to any of its interior points.

Let us consider the following two assertions:

1. Let  $K$  be a compact convex set in the plane and  $K^c = \frac{1}{2}(K - K)$  the centrally symmetrical set of  $K$ . If we consider

$$K_\lambda = \lambda K + (1 - \lambda)K^c$$

then, for all  $0 \leq \lambda \leq 1$ , the convex set  $K_\lambda$  has the same width and diameter as  $K$ ; see (Hernández Cifre, 2000).

2. If  $K$  is a centrally symmetric convex set, then  $K$  can be placed into the circular symmetric slice,  $K_s$ , with the same width and diameter as  $K$ . Let

$$K_\lambda = \lambda K + (1 - \lambda)K_s.$$

Thus, for all  $0 \leq \lambda \leq 1$ , the convex set  $K_\lambda$  has the same width and diameter as  $K$ .

Therefore, it is easy to find examples of convex sets which are mapped into any of the interior points of  $\mathcal{D}$ .

Now, we can assure that inequalities (5), (7), (8), (9) and (10) determine a *complete system of inequalities* for the case  $(A, d, \omega)$ .

#### 4 New solutions to Santaló's problem and the open cases

Besides determining six complete systems of inequalities, in (Santaló, 1961) he conjectured also the possible solution of the cases  $(d, \omega, R)$  and  $(\omega, R, r)$ . Confirmation of both conjectures was obtained later in (Hernández Cifre and Segura Gomis, 2000).

Recently, the cases  $(d, \omega, r)$ ,  $(A, d, R)$ ,  $(p, d, R)$ ,  $(A, R, r)$  and  $(p, R, r)$  have also been solved in (Hernández Cifre, 2000), (Hernández Cifre, 2002) and (Böröczky, Hernández Cifre, and Salinas, 2002).

So, there are still seven open cases for Santaló's problem:  $(A, p, d)$ ,  $(A, d, r)$ ,  $(A, \omega, R)$ ,  $(A, \omega, r)$ ,  $(p, d, r)$ ,  $(p, \omega, R)$  and  $(p, \omega, r)$ .

As we can notice from the above arguments, the problem of determining a complete system of inequalities can be seen, to some extent, as the problem of finding the inequalities that maximize and minimize a particular magnitude for given values of the other measures.

About the last seven cases, some partial results have been obtained (Hernández Cifre and Salinas, 2002). The following table shows briefly, for the remaining open problems, the known solutions in each case and where the gaps are (for the sake of brevity we express the solutions in terms of the extremal sets, instead of giving the corresponding inequalities).

Table 3: The open problems.

Measures	Maximal Sets	Minimal Sets
$(A, p, d)$	Symmetric Lens	Isosceles Triangles + ?
$(A, d, r)$	?	Cap-Bodies
$(p, d, r)$	?	Cap-Bodies
$(A, \omega, R)$	Circular Symmetric Slices	Isosceles Triangles + ?
$(p, \omega, R)$	Circular Symmetric Slices	?
$(A, \omega, r)$	Isosceles Triangles	?
$(p, \omega, r)$	Isosceles Triangles	?

A good reference for this kind of problems is (Salinas, 2002).

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