Total absolute curvatures in spheres via integral geometry

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Abstract

We define total absolute curvatures of compact submanifolds immersed in a sphere from the integral geometry viewpoint. Afterwards, we relate these curvatures to the 'Gauss' maps and level functions defined on the submanifold and we obtain the three different definitions of total absolute curvatures of immersed submanifolds of spheres which appear in the literature. Finally we give an idea of the stereological applications of these definitions.

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1 Introduction

Total curvatures and total absolute curvatures of immersed submanifolds in another Riemannian manifold, usually eucliden space, have been studied in the literature from two points of view.

Chern and Lashof (1957, 1958), defined the total absolute curvatures of immersed submanifolds in the euclidean space by integration, on the submanifold, the absolute value of certain local curvatures. Their work, which relates the theory of total curvatures with Morse theory of critical points of functions defined over the submanifold, has been extended for immersions into spaces of constant curvature and for holomorphic immersions into complex projective spaces; see Teufel (1982) and Ishihara (1986), respectively. Other interesting definitions of total curvatures in different symmetric spaces of rank one, following the idea of Chern and Lashof, can be found in Chih-chy Fwu (1980) and Weiner (1974).

On the other hand, using techniques of integral geometry, Santaló (1969, 1970, 1974) introduced some global definitions of total absolute curvatures for compact manifolds immersed in a euclidean space. Moreover, he showed that one of these total curvatures coincides with the Chern-Lashof curvature. These global definitions have been adapted for immersions into complex spaces by Gual (1992) and Gual and Naveira (1995).

The main purpose of this paper is to use techniques of integral geometry to introduce total absolute curvatures of compact submanifolds immersed in the sphere S^n. The idea is to define these total absolute curvatures in terms of what happens in an $(n-1)$ dimensional sphere or in an $(n-1)$ dimensional 'small' sphere that sweeps through the submanifold; a very convenient property in stereology. Afterwards, we prove that, when spheres $S^{n-1}_{e_0}$ through a fixed point are considered, we obtain the total curvatures defined in Weiner (1974), whereas if we consider spheres S^{n-1} we obtain the total curvatures defined in Teufel (1982). On the other hand, the definition of total absolute curvatures, using parallel $(n-1)$ dimensional 'small' spheres will be the same as in Chih-chy Fwu (1980).

2 Integral geometry: Definition of total absolute curvatures

Let S^n be a unit sphere and $I\!R^{n+1}$ the euclidean vector space with scalar product $< x, y >$. Let M be a compact $m-$dimensional immersed submanifold in S^n and N its unit normal bundle with projection π.

Given an orthonormal frame field $F = \{e_0, e_1, \ldots, e_n\}$ in $I\!R^{n+1}$, we suppose that $e_0 \in S^n$ and that $\{e_1, \ldots, e_n\}$ span $T_{e_0}S^n$.

The linear differential forms ω_{ab} are introduced in Santaló (1976 p.300) by the equations,

$$de_a = \sum_{b=0}^{n} \omega_{ab} e_b, \quad \text{i.e.} \quad w_{ab} =< de_a, e_b > \quad \text{where} \quad \omega_{ab} + \omega_{ba} = 0. \tag{1}$$

There is a one-one correspondence between the Grasmann manifold, $G_{2,n+1}$, of two-dimensional linear subspaces in $I\!R^{n+1}$ and the set of geodesics in S^n. Therefore, the density for all geodesics in S^n is given in Santaló (1976 p.305) by,

$$dS^1 = \bigwedge \omega_{i\alpha}, \qquad i = 0, 1, \quad \alpha = 2, \ldots, n, \tag{2}$$

and the density for geodesics through a fixed point e_0 is given by

$$dS^1_{e_0} = \bigwedge \omega_{1\alpha}, \qquad \alpha = 2, \ldots, n. \tag{3}$$

Let S^1 be a geodesic which is given by a two-dimensional subspace L in $G_{2,n+1}$; then, an hypersphere S^{n-1} in S^n is orthogonal to S^1 if it is given by an $n-$dimensional subspace in $I\!R^{n+1}$, perpendicular to L, which contains the orthogonal complement of L, $L^\perp$. The density for hyperspheres S^{n-1} in S^n is given by

$$dS^{n-1} = \bigwedge \omega_{0\alpha}, \qquad \alpha = 1, \ldots, n, \tag{4}$$

and it coincides with the volume element of S^n at e_0.

Now, we will define three different total absolute curvatures of M using the densities introduced in (2), (3) and (4).

Definition 1 *Let $S^1_{e_0}$ be a geodesic in S^n through a fixed point e_0 and μ_1 the number of hyperspheres S^{n-1}, orthogonal to $S^1_{e_0}$, such that there exists some point $p \in M$ with $T_pM \subset T_pS^{n-1}$, $(p \in S^{n-1})$. We define the total absolute curvature of M with respect to e_0, $\bar{K}_{e_0}(M)$, as the mean value of μ_1 for all $S^1_{e_0}$; that is,*

$$\bar{K}_{e_0}(M) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{G_{1,n}} \mu_1 \, dS^1_{e_0}. \tag{5}$$

Note that the volume of the Grassmannian $G_{1,n}$, (non-oriented geodesics in S^n through a fixed point), is half the surface area of the unit sphere S^{n-1}.

Definition 2 *Let S^1 be a geodesic in S^n and μ_2 the number of hyperspheres S^{n-1}, orthogonal to S^1, such that there exists some point $p \in M$ with $T_pM \subset T_pS^{n-1}$, $(p \in S^{n-1})$. We define the total absolute curvature of M, $\bar{K}(M)$, as the mean value of μ_2 for all S^1 in $G(2, n+1)$; that is,*

$$\bar{K}(M) = \frac{(n-1)!}{(2\pi)^{n-1}} \int_{G_{2,n+1}} \mu_2 \, dS^1. \tag{6}$$

We say that a submanifold Σ^m of S^n is a small $m-$sphere if for any imbedding of S^n into $I\!R^{n+1}$ we have $\Sigma^m = S^n \cap \Pi^{m+1}$, where Π^{m+1} is an $(m+1)$-plane in $I\!R^{n+1}$. If S^{n-1} is an hypersphere in S^n, generated by a subspace L^n in $I\!R^{n+1}$, the small $(n-1)-$spheres parallel to S^{n-1} are generated by n-planes in $I\!R^{n+1}$ parallel to L^n.

Definition 3 *Let S^{n-1} be an hypersphere in S^n and μ_3 the number of small hyperspheres Σ^{n-1}, parallel to S^{n-1}, such that there exists some point $p \in M$ with $T_pM \subset T_p\Sigma^{n-1}$, $(p \in \Sigma^{n-1})$. We define the total absolute curvature of M, $\bar{K}_s(M)$, as the mean value of μ_3 for all S^{n-1}; that is,*

$$\bar{K}_s(M) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{G_{1,n+1}} \mu_3 \, dS^{n-1}. \tag{7}$$

3 Local representation. 'Gauss' maps and Level functions

In this section we will give a local interpretation of the total absolute curvatures defined in (5), (6) and (7), and we will prove that they coincide with the definitions given in Weiner (1974), Teufel (1982) and Chih-chy Fwu (1980), respectively.

We will begin with the total absolute curvature $\bar{K}_{e_0}(M)$; so, we will recall the definition of 'Gauss' map based at e_0 given in Weiner (1974).

Definition 4 *Let $S_{e_0}S^n$ be the unit sphere in the tangent space $T_{e_0}S^n$. We define the 'Gauss' map based at e_0,*

$$\nu_{e_0} : N \longrightarrow S_{e_0}S^n, \tag{8}$$

as follows: let $v \in N_p$; if $p \neq -e_0$, (p is not the antipodal point of e_0), $\nu_{e_0}(v)$ is the parallel translate of v to e_0 along any geodesic from p to e_0; if $p = -e_0$, $\nu_{e_0}(v)$ is the parallel translate of v to e_0 along any geodesic with initial velocity in T_pM.

From the above definition we will give another interpretation for the total absolute curvature of M based at e_0, which appears in Weiner (1974).

Proposition 1 *The total absolute curvature $\bar{K}_{e_0}(M)$ is given by*

$$\bar{K}_{e_0}(M) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_N |\nu_{e_0}^*(dS_{e_0}^1)|. \tag{9}$$

Proof.- Note that expression (9) means the average number of times any vector in $S_{e_0}S^n$ in taken on by ν_{e_0}; therefore, the density $dS_{e_0}^1$ coincides with the volume element of $S_{e_0}S^n$, and the factor 2 on the denominator comes from $v \in N$ implies $-v \in N$.

Let $e \in S_{e_0}S^n$ and $S_{e_0}^1$ the geodesic through e_0 given by e. Let $v \in N_p$ be a vector such that $\nu_{e_0}(v) = e$. Then, if L denotes the subspace generated by p and v, the hypersphere S^{n-1} given by the orthonormal subspace to L with $p \in S^{n-1}$ satisfies that $T_pM \subset T_pS^{n-1}$ and $S^{n-1} \perp S_{e_0}^1$.

On the other hand, given a geodesic $S_{e_0}^1$ in S^n, with initial velocity e, and S^{n-1} an hypersphere such that $T_pM \subset T_pS^{n-1}$ and $S^{n-1} \perp S_{e_0}^1$; the image under ν_{e_0} of the unitary vector $v \perp T_pS^{n-1}$ is e. $\square$

From (9) we define a total (non absolute) curvature of M with respect to e_0 as:

$$K_{e_0}(M) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_N \nu_{e_0}^*(dS_{e_0}^1). \tag{10}$$

Now, we will use the following definition of 'Gauss' map given in Teufel (1982) to rewrite the total absolute curvature $\bar{K}(M)$.

Definition 5 *The 'Gauss' map of M,*

$$\nu : N \longrightarrow S^n,$$

assigns to each $v \in N_p$*, the point of* S^n *obtained by the following inclusions respectively identification:*

$$N_p \subset S_p S^n \subset S_p \mathbb{R}^{n+1} \approxeq S^n. \tag{11}$$

From the above definition, the total absolute curvature, $\bar{K}(M)$, can be expressed as follows.

Proposition 2 *Given a geodesic* S^1 *in* S^n*, we denote by* β *the number of* $v \in N$ *such that* $\nu(v) \in S^1$*. Then,*

$$\bar{K}(M) = \frac{(n-1)!}{2(2\pi)^{n-1}} \int_{G_{2,n+1}} \beta dS^1. \tag{12}$$

Proof.- As it is pointed out in Teufel, (1982, p. 478), the definition of ν shows that $\nu(v) \in S^1$ iff there exists an hypersphere which is tangent to M and normal to v in $\pi(v)$. Therefore, in general, $\beta = 2\mu_2$, where the factor 2 on the right-hand side comes from $v \in N$ implies $-v \in N$. $\square$

Given a geodesic $S^1 = S^n \cap L$, $L \in G_{2,n+1}$, the hyperspheres of S^n orthogonal to S^1 have a common $(n-2)$ sphere $L^\perp \cap S^n$, and they define a level function

$$h_L : S^n - (L^\perp \cap S^n) \longrightarrow S^1. \tag{13}$$

Then, β is the number of critical points of h_L; Teufel (1982, p. 477).

A total curvature of M is defined as

$$K(M) = \frac{(n-1)!}{(2\pi)^{n-1}} \int_{G_{2,n+1}} \sum_{k=0}^{m} (-1)^k \beta_k dS^1, \tag{14}$$

where β_k is the number of critical points of h_L with index k.

Moreover, up to a constant factor, the total curvatures of M are given by $\bar{K}(M) = \int_N |G| dN$ and $K(M) = \int_N G dN$, where $G(v)$ is the determinand of the second fundamental tensor $S(v) : T_{\pi(v)}M \longrightarrow T_{\pi(v)}M$.

Finally, we show that the total absolute curvature $\bar{K}_s(M)$ given in (7) appears in Chih-chy Fwu (1980), where it is defined using the following definition of 'level' function.

Definition 6 *Given a point* e_0 *in* S^n*, we define the height function* $h_{e_0} : M \longrightarrow \mathbb{R}$ *by* $h_{e_0}(p) = < e_0, p >$*, where* p *is considered a vector in* $\mathbb{R}^{n+1}$ *from the composite immersion of* M *in* S^n *and the standard immersion of* S^n *in* $\mathbb{R}^{n+1}$*.*

Proposition 3 *Let S^{n-1} be an hypersphere in S^n generated by a subspace L^n in $\mathbb{R}^{n+1}$, and e_0 the unit vector which generates the orthogonal complement to L^n. Then,*

$$\bar{K}_s(M) = \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}} \int_{S^n} \gamma dS, \tag{15}$$

where γ is the number of critical points of the height function h_{e_0}.

Proof.- If there exists a small hypersurface Σ^{n-1}, parallel to S^{n-1}, such that there exists a point $p \in M$ with $T_pM \subset T_p\Sigma^{n-1}$, and v denotes the normal vector to Σ^{n-1} at p. Then, $v \in N_p$ and $e_0 = \cos(a)p + \sin(a)v$, $(a \in [0, \pi])$ (i.e. p is a critical point of h_{e_0}, Chih-chy Fwu (1980)); and vice versa. □

As in the preceding case, another total curvature of M, $K_s(M)$, is given by

$$K_s(M) = \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}} \int_{S^n} \sum_{k=0}^{m} (-1)^k \gamma_k dS, \tag{16}$$

where γ_k is the number of critical points of h_{e_0} with index k.

The total curvatures defined in (10) (when $-e_0 \notin M$) and (16) give the Euler-Poincaré characteristic of M; (see Weiner (1974) and Chih-chy Fwu (1980), respectively.)

Definitions 1 and 2 allow us to obtain the total absolute curvatures of M counting the number of points $p \in M$ where there exits an hypersphere tangent to M at p and orthogonal to a given geodesic; whereas from Definition 3 it is possible to obtain another total absolute curvature counting the number of points of M where there exits an small hypersphere tangent to M and parallel to a given hypersphere. These are interesting properties in stereology because we obtain information of a submanifold in S^n in terms of what happens in an $(n-1)-$dimensional sphere (or $(n-1)-$dimensional small sphere) that sweeps through the submanifold.

For instance, if the parameter to estimate is the total absolute curvature of ∂D, where D is a domain with boundary in S^2; i. e.

$$Q = \int_{\partial D} |k_g| ds, \tag{17}$$

where k_g is the geodesic curvature of ∂D, since, Definition 2, gives, up to a constant factor, the value of Q (see Teufel (1986)), and $G_{2,3}$ can be identified with half an sphere, it is possible to estimate Q from systematic sampling on the sphere, Gual-Arnau and Cruz-Orive (2000).

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